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ON SOME RECIPROCAL RELATIONS IN THE THEORY OF NONSTATIONARY FLOWS

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SUMMARY

In the theory of nonstationary flows about airfoils, the "indicial lift" function $k_1(s)$ of Wagner and the "alternating lift" function $C(k)$ of Theodorsen have fundamental significance. This paper reports on some interesting relations of the nature of Fourier transforms that exist between these functions. General problems in transient flows about airfoils may be given a unified broad treatment when these functions are employed. Certain approximate results also are reported which are of notable simplicity, and an analogy with transient electrical flows is drawn.

INTRODUCTION

There exist at the present time two significant functions that have been introduced into the two-dimensional potential theory of nonuniform motion of airfoils, one by Wagner (reference 1) and the other by Theodorsen (reference 2). Wagner's function concerns the growth of circulation or lift about an airfoil at a small fixed angle of attack starting impulsively from rest to a uniform velocity v. Theodorsen's function describes the lift due to circulation about an airfoil oscillating sinusoidally and moving with uniform velocity v. It is the object of this paper to note the usefulness of these functions in handling a wide class of problems in transient flows about airfoils and to point out certain interesting relations existing between them. These relations are of the nature of Fourier transforms, which occur with remarkable abundance in numerous fields of physics and which are one of the main studies of the recently popular operational-calculus methods. A noteworthy analogy between transient hydrodynamic flows and transient electrical flows is also mentioned.

A third interesting function, which concerns the behavior of an airfoil upon entering a gust, has been introduced by Küssner in reference 3, which is an excellent survey of the status of the problem of nonstationary flows about airfoils. This function and its relation to Wagner's function will also be discussed.

THEORY OF NONSTATIONARY FLOWS

Wagner's function, $k_1(s)$.—Let the chord of the airfoil be $2b$ and let the angle of attack (assumed small) be α. Also let the impulsive motion from rest to uniform velocity v take place at the origin, $s=0$ (fig. 1). The vertical velocity at the airfoil surface is $w=v \sin\alpha$.

Then, based on the physical assumption that the velocity at the trailing edge is finite for all time, Wagner derives for the lift, as a function of $s=vt/b$,

$$L=2\pi b \rho v w k_1(s) \qquad (1)$$

The function $k_1(s)$ is illustrated in figure 2. Wagner does not derive an explicit analytic expression for $k_1(s)$ but gives only numerical values. Küssner (reference 3) derives a slowly convergent expression in series form for $k_1(s)$ that checks Wagner's values. The expression is rather long and will not be reproduced
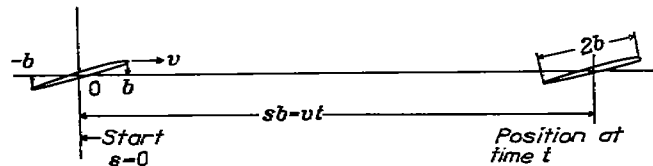
FIGURE 1.—Illustration of nondimensional parameter s, distance traversed in terms of half chord b.

here. It is of great interest to note that the following simple expression

$$k_1(s)\cong 1-\frac{2}{4+s} \qquad (1a)$$

agrees within 2 percent in the entire range $0<s<\infty$. (Cf. table I.) This expression, which may be regarded as a fortunate choice of the author, is especially good
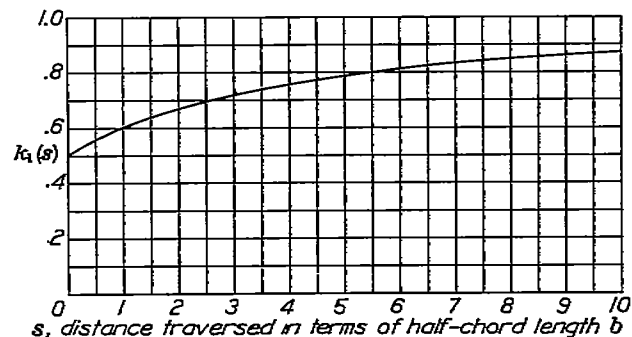
FIGURE 2.—The function $k_1(s)$ of Wagner.

in the range s small; it may be reasoned that it represents the actual physical state more closely than the theoretical solution, since it approaches the steady condition $(s=\infty)$ somewhat more slowly.

It is observed that half the final lift is assumed at once and that the lift gradually approaches its asymptote, $2\pi b \rho vw$, in agreement with the results for stationary flows.

The function $k_1(s)$ is analogous to what is termed in electrical-circuit theory the "indicial-admittance function" $A(t)$, which is the current response of a linear network to a suddenly applied unit voltage; substitute lift for current and unit vertical velocity for unit voltage to complete the analogy.

Theodorsen's function, $C(k)$.—The lift on an airfoil oscillating sinusoidally through a small angle of attack and moving with uniform velocity v is given by the sum of two parts: (a) A classical noncirculatory part and (b) a part due to circulation. This paper is not concerned with the classical part, which consists of the virtual inertia terms, the general concepts of which go back to the time of Kirchoff and Kelvin.

The steady-state part of the lift due to circulation about an oscillating airfoil moving with velocity v has been given by Theodorsen as

$$L=2\pi b\rho vC(k)Q \tag{2}$$

where

$Q=we^{i\omega t}$ is the vertical velocity at the three-quarter chord point [1] in complex form,

and where

$k=\dfrac{\omega b}{v}$, where ω is the angular frequency.

The parameter k permits the substitution of distance s in place of time t since $\omega t=\dfrac{\omega b}{v}\dfrac{vt}{b}=ks$. The function $C(k)$ is defined as

$$C(k)=\frac{H_1^{(2)}}{H_1^{(2)}+iH_0^{(2)}} \tag{3}$$

where $H_0^{(2)}$ and $H_1^{(2)}$ are Hankel functions (Bessel functions of the third kind), or, separated into real and imaginary parts and expressed in terms of Bessel functions of the first and second kinds,

$$C(k)=F(k)+iG(k) \tag{4}$$

where

$$F(k)=\frac{J_1(J_1+Y_0)+Y_1(Y_1-J_0)}{(J_1+Y_0)^2+(Y_1-J_0)^2}$$

$$G(k)=-\frac{Y_1Y_0+J_1J_0}{(J_1+Y_0)^2+(Y_1-J_0)^2}$$

These functions are illustrated in two ways in figure 3.

It is important to note that, in the interpretation of equation (2), $C(k)$ is considered to operate on the function Q. Thus, suppose the actual vertical velocity is $w_0 \sin ks$. This quantity must be expressed as I. P. w_0e^{iks}. Then the lift is

I. P. $2\pi b\rho vw_0C(k)e^{iks}$

or

$$2\pi b\rho vw_0(F^2+G^2)^{\frac{1}{2}}\sin\left(ks+\tan^{-1}\frac{G}{F}\right)$$

The lift thus has the same frequency as the vertical velocity and both its magnitude and phase are functions of k.

The analogy with alternating currents in electrical networks can be mentioned. The function $C(k)$ corresponds to the complex admittance function for alternating currents (reciprocal of the complex impedance function $Z(\omega)$). The real and imaginary components F and G are analogous, respectively, to the alternating-current power and quadrature components; the lift due to complex vertical velocity Q corresponds to current due to complex voltage E.
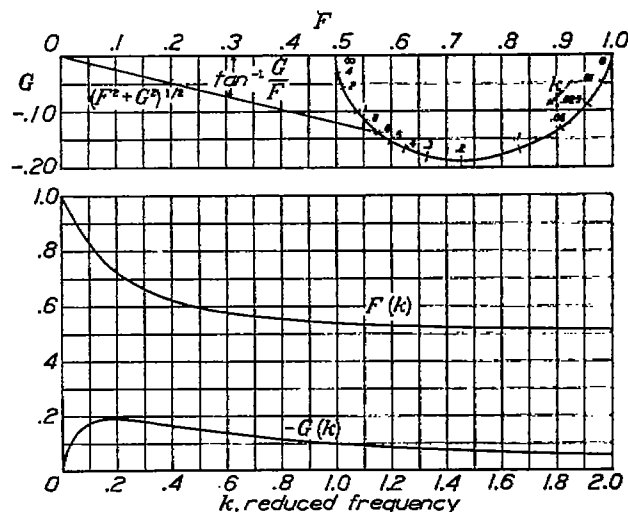

FIGURE 3.—The function $C(k)=F+iG$ of Theodorsen.

Superposition principle.—Linearity of the equations of Wagner permits one to write, in general, for the lift due to a vertical-velocity function $w(s)$ (at the three-quarter chord point) suddenly applied at the instant when $s=0$,

$$\frac{L}{2\pi b\rho v}=w(0)k_1(s)+\int_0^s k_1(s-s_1)\frac{dw}{ds_1}ds_1 \tag{5}$$

where $w(0)$ is the value of w at $s=0$. This result can be derived by replacing the function $w(s)$ by a step function the envelope of which is $w(s)$ and by going to the limit. The equation is a well-known mathematical result and is often employed in electrical-circuit theory (reference 4, p. 68).

Several useful forms of this equation exist, of which one is

$$\frac{L}{2\pi b\rho v}=w(0)k_1(s)+\int_0^s k_1(s_1)\frac{d}{ds_1}w(s-s_1)ds_1 \tag{5a}$$

Equation (5) expresses a noteworthy result, since it permits the handling (at least formally or graphically) of many problems in transient flows that would otherwise be quite laborious. As an example of its application, and as a means of obtaining an interesting result, let equation (5a) be applied to the vertical-velocity function of the form $w(s)=w_0e^{iks}$.

There results for the lift, when the vertical velocity $w_0 e^{iks}$ is suddenly applied at $s=0$,

$$\frac{L}{2\pi b\rho v}=w_0 k_1(s)+w_0 ike^{iks}\int_0^s k_1(s_1)e^{-iks_1}ds_1 \qquad (5b)$$

In order to isolate the transient and steady-state parts, the familiar device of separating the interval 0 to s into an interval 0 to ∞ minus an interval s to ∞ is used.

Then, for the steady-state part only (writing s in place of s_1 in the definite integral),

$$L=2\pi b\rho v w_0 ike^{iks}\int_0^\infty k_1(s)e^{-iks}ds$$

This equation must agree with Theodorsen's result (equation (2)), which may be written in this case

$$L=2\pi b\rho v w_0 C(k)e^{iks}$$

Hence it must follow that

$$C(k)=ik\int_0^\infty k_1(s)e^{-iks}ds \qquad (6)$$

or, in better mathematical form,

$$C(k)-1=ik\int_0^\infty [k_1(s)-1]e^{-iks}ds \qquad (6a)$$

It follows, for the components of $C(k)=F+iG$, that

$$\frac{F(k)}{k}=\int_0^\infty k_1(s)\ \sin ks\ ds \qquad (7)$$

$$\frac{G(k)}{k}=\int_0^\infty [k_1(s)-1]\ \cos ks\ ds \qquad (8)$$

These equations can be inverted by the properties of Fourier transforms (reference 4, p. 183). Then

$$k_1(s)=\frac{2}{\pi}\int_0^\infty \frac{F(k)}{k}\ \sin ks\ dk \qquad (9)$$

$$=1+\frac{2}{\pi}\int_0^\infty \frac{G(k)}{k}\ \cos ks\ dk \qquad (10)$$

Some approximate results.—The integration expressed in equation (6) can be performed directly when the approximate expression $k_1(s)\cong 1-\frac{2}{4+s}$ is employed. Then the following simpler expressions for $C(k)$, F, and G, which hold within a few percent, are obtained (cf. Jahnke-Emde, "Tables of Functions," p. 80): [2]

$$C(k)-1\cong 2ike^{4ik}\mathrm{Ei}(-4ik) \qquad (6b)$$

$$\frac{F-1}{2k}\cong \cos 4k\ \mathrm{si}4k-\sin 4k\ \mathrm{Ci}4k$$

$$\frac{G}{2k}\cong \cos 4k\ \mathrm{Ci}4k+\sin 4k\ \mathrm{si}4k$$

The following approximate result is also of interest. Let the vertical-velocity function be of the form $w(s)=w_0(1-e^{-\gamma s})$ where γ determines the rate at which $w(s)$ approaches w_0. Then from (5), using the approximate expression for $k_1(s)$,

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$$\frac{L}{2\pi b\rho v w_0}=1-e^{-\gamma s}-2\gamma e^{-(4+s)\gamma}[\mathrm{Ei}(4+s)\gamma-\mathrm{Ei}(4\gamma)]$$

Second derivation of equation (6).—The result expressed in equation (6) can be demonstrated in still another way, which, essentially, is Küssner's treatment. (Cf. reference 3, p. 420.) A common artifice in the treatment of unit discontinuities is to represent the unit "jump" function

$$1(s)=1 \text{ for } s>0$$
$$1(s)=0 \text{ for } s<0$$

by the following integral in the complex k plane, which can be evaluated by residue theory:

$$1(s)=\frac{1}{2\pi i}\int_{\curvearrowright} \frac{e^{iks}}{k}dk$$

Here the hook integral means integrate from $-\infty$ to $+\infty$, bypassing the singular point at the origin by a small semicircle from below. In effect, a spectrum analysis
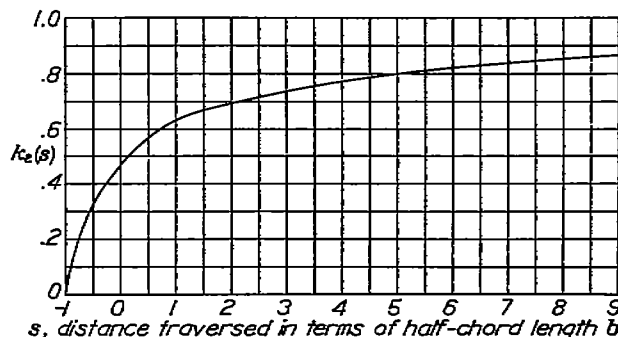

FIGURE 4.—The function $k_2(s)$ of Küssner.

of the unit jump function has been made as a limit of a sum of exponential terms. To each exponential term of vertical velocity there corresponds the lift given by multiplication with Theodorsen's function $C(k)$. Then,[3] by addition

$$k_1(s)=\frac{1}{2\pi i}\int_{\curvearrowright} \frac{C(k)}{k}e^{iks}dk$$

$$=1+\frac{1}{2\pi i}\int_{-\infty}^\infty \frac{C(k)-1}{k}e^{iks}\ dk\ [s>0]$$

$$=0\ [s<0]$$

Separating $C(k)$ into $F(k)+iG(k)$ and noting that $C(-k)=F(k)-iG(k)$, there results

$$k_1(s)=1+\frac{2}{\pi}\int_0^\infty \frac{G(k)}{k}\cos ks\ dk$$

$$=\frac{2}{\pi}\int_0^\infty \frac{F(k)}{k}\sin ks\ dk$$

which check equations (9) and (10) and hence also equations (7) and (8).

Küssner's function, $k_2(s)$.—Küssner has derived a function $k_2(s)$ (fig. 4), which gives the lift on an airfoil as it *penetrates* into a sudden vertical-gust region without change in direction. If the change in vertical

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[1] The expression (6b) may be considered the limit, as $s\to\infty$, of the function (cf. equation (5b)):

$$C(k,s)-1\cong 2ike^{4ik}\{\mathrm{Ei}(-4ik)-\mathrm{Ei}[-(4+s)\ ik]\}$$

The transient term containing s approaches 0 as $s\to\infty$.

[2] The relation between $C(k)$ and $k_1(s)$ is expressed in the operational calculus as $C(k)1(s)=k_1(s)$, where ik is the operator d/ds and the expression is interpreted "the function $C(k)$ operating on the unit jump function $1(s)$."

velocity in the gust region is w (assumed constant), the change in lift as the airfoil leading edge penetrates the gust region is

$$L = 2\pi b \rho v w k_2(s) \tag{11}$$

Note that the function $k_2(s)$ is defined for $s > -1$, and is 0 for $s < -1$, i. e., before the leading edge has penetrated the gust region.

In order to obtain the relation of $k_2(s)$ to $k_1(s)$, note that the operational equivalent of $k_2(s)$ (in the same sense[4] that $C(k)1(s) = k_1(s)$) is $C(k)$ $[J_0(k) + iJ_1(k)]$ (reference 3, p. 420). This latter function describes the steady-state lift due to circulation on an airfoil moving with uniform velocity v and whose vertical velocity is oscillating sinusoidally but progressing in the form of waves from point to point, i. e., the vertical velocity is of the form

$$w_0 e^{i(\omega t + kx)} = w_0 e^{ik(vt+x)}$$

where x defines any point of the airfoil measured from the center.

Then, the following relations hold, writing σ for $s+1$,

$$C(k)\ 1(\sigma) = k_1(\sigma)$$

$$[J_0(k) + iJ_1(k)]\ 1(\sigma) = S(\sigma) = \frac{1}{2} + \frac{1}{\pi} \arcsin{(\sigma - 1)}$$

$$+ \frac{1}{\pi}\sqrt{1 - (\sigma - 1)^2}\ [0 < \sigma < 2]$$

$$= 1\ [\sigma > 2]$$

The second relation is given by Nielsen in "Handbuch der Cylinderfunctionen," page 197. By superposition, there results for the combined operator

$$C(k)\ [J_0(k) + iJ_1(k)]\ 1(\sigma),$$

where

$$k_2(s) = \int_0^\sigma k_1(\sigma - \lambda) S'(\lambda)\, d\lambda$$

where

$$S'(\lambda) = \frac{1}{\pi}\sqrt{\frac{2 - \lambda}{\lambda}}\ [0 < \lambda < 2]$$

$$= 0\ [\lambda > 2]$$

Hence,

$$k_2(s) = \frac{1}{\pi}\int_0^\sigma k_1(\sigma - \lambda)\sqrt{\frac{2 - \lambda}{\lambda}}\, d\lambda\ [0 < \sigma < 2] \tag{12}$$

$$= \frac{1}{\pi}\int_0^2 k_1(\sigma - \lambda)\sqrt{\frac{2 - \lambda}{\lambda}}\, d\lambda\ [\sigma > 2]$$

or, expressed in terms of s,

$$k_2(s) = \frac{1}{\pi}\int_{-1}^s k_1(s - s_1)\sqrt{\frac{1 - s_1}{1 + s_1}}\, ds_1\ [-1 < s < 1]$$

$$= \frac{1}{\pi}\int_{-1}^1 k_1(s - s_1)\sqrt{\frac{1 - s_1}{1 + s_1}}\, ds_1\ [s > 1] \tag{12a}$$

The effect of an arbitrary gust function $w(\sigma)$ can be written directly by superposition $(\sigma = s+1)$

$$\frac{L}{2\pi b \rho v} = w(0) k_2(\sigma) + \int_0^\sigma k_2(\sigma_1)\frac{d}{d\sigma_1} w(\sigma - \sigma_1)\, d\sigma_1 \tag{13}$$

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[4] For a correction to an error in sign that exists in reference 3, p. 420, consult reference 5. The values of k²(s) here given therefore need to be modified; the corrected values, including the apparent-mass effect due to change of shape, are presented in reference 5.

The approximate expression $k_1(s) \cong 1 - \dfrac{2}{4+s}$ may be put into equation (12). Then (cf. table I),

$$k_2(\sigma) \cong 2\sqrt{\frac{2+\sigma}{4+\sigma}} - 1\ [\sigma > 2]$$

$$\cong \frac{1}{\pi}\left[\sqrt{\sigma(2 - \sigma)} - \cos^{-1}(1 - \sigma)\right.$$

$$\left. + 4\sqrt{\frac{2+\sigma}{4+\sigma}}\sin^{-1}\sqrt{\frac{\sigma(2+\sigma)}{8}}\right]\ [\sigma < 2]$$
$$\tag{12b}$$

CONCLUDING REMARKS

It has been shown that the functions $C(k)$ and $k_1(s)$ are of considerable significance in the theory of non-stationary flows. To a certain extent, the results are formal since many of the analytic properties of these functions lie hidden in their complicated structures. Further mathematical studies of the function $C(k)$ when k is a complex variable would appear to be desirable, since this function is associated with the lift due to a general damped sinusoidal motion of the airfoil. In many problems in transient flows, it is therefore of value to employ the approximate expressions for $C(k)$, $k_1(s)$, and $k_2(s)$ given and thus to obtain quickly a simpler perspective of the problem.

LANGLEY MEMORIAL AERONAUTICAL LABORATORY,
    NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
    LANGLEY FIELD, VA., *March 28, 1938.*

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TABLE I.—VALUES OF $k_1(s)$ AND $k_2(s)$

s	¹ $k_1(s)$	² $k_1(s)$approx.	³ $k_2(s)$	⁴ $k_2(s)$approx.
≤-1	0	0	0	0
-.5	0		.328	.328
0	.5000	.5000	.468	.468
.5	.5557	.5556	.568	.568
1	.6006	.6000	.631	.633
2	.6693	.6667	.693	.690
3	.7195	.7143	.737	.732
4	.7582	.7500	.772	.764
5	.7880	.7778	.799	.789
6	.8125	.8000	.822	.809
7	.8325	.8182	.840	.826
8	.8485	.8333	.855	.840
9	.8625	.8461	.868	.852
10	.8745	.8572	.877	.860
20	.9321	.9107	.934	.913
∞	1.0000	1.0000	1.000	1.000

¹ Values of $k_1(s)$ taken from references 1 and 3.
² From equation (1a).
³ Values of $k_2(s)$ calculated from equation (12).
⁴ From equation (12b).